The Transfer Evolution Formalism : an introduction to model partitioning and coupling, and feedback analysis

The TEF-ZOOM Collaboration*
(CIRED†, LE‡, LMD§)

20 septembre 2005

The purpose of the Transfer Evolution Formalism, from its genesis, is to allow the modelling of complex systems in which various involved processes are more or less strongly coupled. The analysis of the coupling mechanisms which are influencing the behaviour of the different parts of the model is fundamental, as it allows some understanding of the links between elementary processes that are to be modelled. The interaction between sub-models which may be "black boxes" needs specific methods to drive the analysis of the imbedded couplings and feedbacks.

The modelling procedure should meet several requirements :

- modularity to allow the implementation, in one single model, of various sub-models suited to each process or each part of the system,
- able to provide information on the couplings between the various evolutions of different parts of the system,
- it must allow to study the non-stationary dynamics as well as the search of stationary solutions.

The description of a system using a modular model will proceed in two complementary steps : partitioning of the global system in different subsystems (corresponding to different phenomena), and connecting of all these parts to recover the global system. To acomplish that feature, One replaces the global system by a set of partial models and (eventually) of additional variables and relations, such that the new set of equations be mathematicaly equivalent to the global model.

*Paper originated by Venance Journé. Final redaction : Jean-Yves Grandpeix, Stéphane Hallegatte, Alain Lahellec and Venance Journé. Reviewers : Stéphane Blanco, Patrice Dumas, Jean-Louis Joly and Vincent Platel.

†Centre International de Recherche sur l'Environnement et le Développement , Paris

‡Laboratoire d'Énergétique, Université Paul Sabatier, Toulouse and its Tarbes antenna

§Laboratoire de Météorologie Dynamique, Paris

For example, in the field of environmental integrated assessment[1], there exist a number of detailed disciplinary models. It is important to be able to study the interfaces between these models and such a formalism as the TEF is well suited to this task. One reason is that the resolution of the system of equations proceeds by the elimination of all variables except the ones describing the interfaces to be analysed.

1 Partitioning and re-coupling a system using the TEF : cells and transfers.

Modelling of a system through partitioning and re-coupling is involved in system analysis, when asking questions as : what cross-influence two subsystems have on one another. To answer such question, each sub-model must be well defined, and an interface between these objects generally needs to be conceived. This is what is called partitioning here.

A TEF partitioning is applied in a recursive manner, each subsystem of a partition being submit to partitioning again. The final elements of the partitioning should represent a satisfying level of analysis of the original system. A good practice considers an elementary object as eventually submit to an experimental procedure, hence quasi isolated from the environment, and linked to it through well identified physical quantities (interface conditions).

Let start with a system defined from its state equations :

$$\frac{d\vec{\eta}}{dt} = \vec{G}(\vec{\eta}, \vec{\psi}, t) \tag{1}$$

Given initial conditions $\vec{\eta}(t_0)$, and boundary conditions, and forcing (independent sources) $\vec{\psi}(t)$, a system trajectory is hence determined[2] : one says that the mathematical system is "well posed".

The **partitioning** is applied to the vector space $\{\vec{\eta}\}$, leading to a state equation for each sub-vector space $\{\vec{\eta_\alpha}\}$:

$$\frac{d\vec{\eta}_\alpha}{dt} = \vec{G}(\vec{\eta}_\alpha, \vec{\varphi}, \vec{\psi}, t) \tag{2}$$

where some new boundary conditions $\vec{\varphi}$ had to be added as needed by the TEF to have well posed sub-systems. Two consequences are drawn :

1. each element, called "a cell", should not explicitly depend on another cell state variables ;
2. the interface that had to be introduced represents **dependant** boundary conditions, which means they depend on cells state.

[1] TEF-ZOOM has been developped in the fields of Engineering Sciences. This text was written for a European Program towards integrated assesment, which explains the choice of some illustrative examples.

[2] between eventual singularities

Suppose these interfacing conditions are chosen, hence each cell will follow a trajectory that we call a "decoupled trajectory". For these trajectories to become the true system trajectory, one needs to specify a supplemental equation for each interface variable which has to explicit how the variable is constrained by the state space environment. The TEF calls "transfers" these interfacing objects :

$$\vec{\varphi} = \vec{f}(\vec{\eta}, \vec{\varphi}, t) \tag{3}$$

Equation (3) expresses that transfer variables are statically determined by state variables, and hence do not introduce new initial conditions. **Coupling** is the operation that defines all transfers between cells. They usually correspond to mathematical constraints (continuity) or physical quantities exchanged between cells (as fluxes).

On a mathematical point of view, systems $(2, 3)$ and (1) have the same solution. To summarise, partitioning and coupling with the TEF uses two classes of objects, and the two classes of associated variables. **Cells** are mathematically represented by state variables and state equations (equations of evolution using partial time derivative), and **transfers** as static constraint equations.

Cells models are very similar to boundary condition problems. The difference lies in the introduction of a dependence of these conditions as being function of the rest of the system. One can infer from the TEF structure of a system that the analysis elements of inter-influence are set : a cell has to go through transfers to influence the rest of the system. Reciprocally, the reacting system retro-influences each cell through the set of transfers connecting the cell to the system.

To fully explicit the mutual influences, the TEF modelling of a cell explicits the sensitivity of the state variables to changing interface conditions. This is mathematically applied through the linearisation of the state space equations, as will now be described.

2 Linearisation, analysis of the TEF operators, and resolution

The relation between sub-systems is excessively difficult to exhibit when having to cope with non-linear systems. To meat that purpose, the TEF uses the Tangent Linear System (TLS) along a trajectory. The continuous time linearised equations are built firstly, and a time scheme is applied in a second phase.

After these two steps, one gets each time step, an algebraic system equivalent to the full system. The linearised system allows analyses in terms of sensitivity to perturbation, coupling matrices or feedback factors. The TEF uses the TLS both for simulation (system trajectory), and analysis.

2.1 Linearisation

One considers the non-linear system $(2, 3)$, over a small portion along its trajectory (say between t_0 and $t_0 + \delta t$). The variation $\overrightarrow{\delta\eta}$ of $\vec{\eta}$, and $\overrightarrow{\delta\varphi}$ of $\vec{\varphi}$ is obtained

through Taylor expansion :

$$\frac{\partial \overrightarrow{\delta\eta}}{\partial t}(\tau) = \vec{G}\Big|_{t_0} + \left.\overline{\overline{\frac{\partial G}{\partial \eta}}}\right|_{t_0} \overrightarrow{\delta\eta}(\tau) + \left.\overline{\overline{\frac{\partial G}{\partial \varphi}}}\right|_{t_0} \overrightarrow{\delta\varphi}(\tau) + \left.\frac{\partial \vec{G}}{\partial t}\right|_{t_0} \tau \tag{4}$$

$$\overrightarrow{\delta\varphi}(\tau) = \left.\overline{\overline{\frac{\partial f}{\partial \eta}}}\right|_{t_0} \overrightarrow{\delta\eta}(\tau) + \left.\overline{\overline{\frac{\partial f}{\partial \varphi}}}\right|_{t_0} \overrightarrow{\delta\varphi}(\tau) + \left.\frac{\partial \vec{f}}{\partial t}\right|_{t_0} \tau \tag{5}$$

One obtains a partial difference equation system called the Tangent Linear System to the trajectory, with $\tau \in [t, t+\delta t[$ parameterizing the trajectory.

2.2 Time scheme

Solving a linear system is classical and may be done using either a time scheme or a linear transformation (Laplace, Fourier, Borel,...). We use here a time scheme to obtain the lumped trajectory. The application of a linear transformation will be used later to introduce the system analysis.

A Crank-Nicolson scheme is applied to system (4,5) to obtain :

$$\begin{cases} \overline{\overline{A}}\overrightarrow{\delta\eta} + \overline{\overline{B}}\overrightarrow{\delta\varphi} = \vec{\Gamma}\delta t \\ \overline{\overline{C^+}}\overrightarrow{\delta\eta} - (\mathbb{I} + \overline{\overline{D}})\overrightarrow{\delta\varphi} = \vec{\Omega}\delta t \end{cases} \tag{6}$$

where the introduced symbols of matrices and vectors are defined by :

$$\begin{cases} \overline{\overline{A}} = \mathbb{I} - \frac{\delta t}{2} \left.\overline{\overline{\frac{\partial G}{\partial \eta}}}\right|_{t_0} \\ \overline{\overline{B}} = -\frac{\delta t}{2} \left.\overline{\overline{\frac{\partial G}{\partial \varphi}}}\right|_{t_0} \\ \vec{\Gamma} = \vec{G}\Big|_{t_0} + \frac{\delta t}{2} \left.\frac{\partial \vec{G}}{\partial t}\right|_{t0} \\ \overline{\overline{C^+}} = \left.\overline{\overline{\frac{\partial f}{\partial \eta}}}\right|_{t_0} \\ \overline{\overline{D}} = -\left.\overline{\overline{\frac{\partial f}{\partial \varphi}}}\right|_{t_0} \\ \vec{\Omega} = -\left.\frac{\partial \vec{f}}{\partial t}\right|_{t_0} \end{cases} \tag{7}$$

The matrix $(\mathbb{I} + \overline{\overline{D}})$ has been introduced because the overall set of transfer equations is also partitioned, so that subsets of transfers can be linked between each-other, contrary to cells.

The preceding formulation is responding to our programme as we now will show.

2.3 Interpreting the elements of the TEF

An ultimate rewriting of system (6) is used to open to an interpretation of the set of mathematical elements introduced by the TEF in modelling the original system. Let eliminate $\overrightarrow{\delta\eta}$ from the second equation of (6) :

$$\left\{ \begin{array}{l} \overrightarrow{\delta\eta} = \overrightarrow{\delta\eta}_{dec} + \overline{\overline{F}}\overrightarrow{\delta\varphi} \\ \left(\mathbb{I} + \overline{\overline{\mathbb{C}_{(\delta t)}}}\right) \overrightarrow{\delta\varphi} = \overrightarrow{\delta\varphi}_{ins} \end{array} \right. \tag{8}$$

The Jacobian matrices in cells are defining $\overline{\overline{F}} = -\overline{\overline{A^{-1}}}\,\overline{\overline{B}}$. This matrix expresses the sensitivity of a cell to change in its interfacing conditions $\vec{\varphi}$. When this matrix is a null matrix, or when $\overrightarrow{\delta\varphi} = 0$, one gets $\overrightarrow{\delta\eta} = \overrightarrow{\delta\eta}_{dec}$, which is the cell evolution when the interface is "frozen". It is called the "decoupled evolution" of the cell (i.e. along the local TLS associated to a time-step [1]) :

$$\overrightarrow{\delta\eta}_{dec} = \overline{\overline{A^{-1}}}\,\vec{\Gamma}\,\delta t$$

In an analog way , $\overline{\overline{C^+}}$ from (6) determines the evolution of the transfers during the time-step while cells are evolving. When $\overrightarrow{\delta\eta}$ is eliminated from the transfer equation, which is always possible[2], one obtains the second line of (8), with :

$$\overline{\overline{\mathbb{C}_{(\delta t)}}} = \overline{\overline{D}} + \overline{\overline{C^+}}\,\overline{\overline{A^{-1}}}\,\overline{\overline{B}}$$

and

$$\overrightarrow{\delta\varphi}_{ins} = \overline{\overline{C^+}}\,\overrightarrow{\delta\eta}_{dec} - \vec{\Omega}\delta t$$

$\overline{\overline{\mathbb{C}_{(\delta t)}}}$ is called a "coupling matrix" between transfers. Under this form, it is depending on δt. We shall review later on how this matrix is important to system analysis.

$\overrightarrow{\delta\varphi}_{ins}$ is the transfers evolution when $\overline{\overline{B}} = 0$. $\overrightarrow{\delta\varphi}_{ins}$ is called the "insensitive transfer evolution" because cells are following their decoupled trajectory. In that case, one notices that $\overline{\overline{\mathbb{C}_{(\delta t)}}} = \overline{\overline{D}}$, which means that cells are not coupling any more transfers among themselves.

To summarise, using the local TLS gives access to a mathematical formulation that provides the full dynamics of a system, allowing to explicit the operators (matrices) which determine the inter-influence between elementary objects. It also allows to define a portion of the trajectory, using the solution of a matrix algebraic system.

2.4 Matricial structure of the system and solution

The matrix structure of system (6) is the following :

$$\left[\begin{array}{cc} \overline{\overline{A}} & \overline{\overline{B}} \\ -\overline{\overline{C^+}} & \mathbb{I} + \overline{\overline{D}} \end{array} \right] \left[\begin{array}{c} \overrightarrow{\delta\eta} \\ \overrightarrow{\delta\varphi} \end{array} \right] = \left[\begin{array}{c} \Gamma\delta t \\ -\Omega\delta t \end{array} \right]$$

[1] more rigorously, in the local differential manifold

[2] being a well-posed problem, $\overline{\overline{A}}$ is regular

The system is solved in a two-step process, eliminating firstly the cell variables using the same procedure as for (8) :

$$\overrightarrow{\delta\eta} = -\overline{\overline{A^{-1}}}\,\overline{\overline{B}}\,\overrightarrow{\delta\varphi} + \overline{\overline{A^{-1}}}\,\vec{\Gamma}\,\delta t$$

secondly, $\overrightarrow{\delta\eta}$ is introduced in the second equation :

$$\overline{\overline{C^+}}\left(-\overline{\overline{A^{-1}}}\,\overline{\overline{B}}\,\overrightarrow{\delta\varphi} + \overline{\overline{A^{-1}}}\vec{\Gamma}\delta t\right) - \left(\mathbb{I} + \overline{\overline{D}}\right)\overrightarrow{\delta\varphi} = \vec{\Omega}\,\delta t$$

$$\left(\mathbb{I} + \overline{\overline{D}} + \overline{\overline{C^+}}\,\overline{\overline{A^{-1}}}\,\overline{\overline{B}}\right)\overrightarrow{\delta\varphi} = \overline{\overline{C^+}}\,\overline{\overline{A^{-1}}}\,\vec{\Gamma}\delta t - \vec{\Omega}\,\delta t$$

This linear system is classically solved to give $\overrightarrow{\delta\varphi}$. Each cell-equation can be solved, as all terms except $\overrightarrow{\delta\eta}$ are known.

2.5 Numerical process

Once given the general procedure of solving a TEF system, a few difficulties may be described. The manipulation of the Jacobian matrices can lead to problems. Let explicit an arbitrary system of three cells labelled (α, β, γ) and the set of all transfers connecting them, as a unique vector $\vec{\varphi_u}$:

$$\begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ u \end{array} \begin{pmatrix} \overline{\overline{A}}_{\alpha\alpha} & 0 & 0 & \overline{\overline{B}}_{\alpha u} \\ 0 & \overline{\overline{A}}_{\beta\beta} & 0 & \overline{\overline{B}}_{\beta u} \\ 0 & 0 & \overline{\overline{A}}_{\gamma\gamma} & \overline{\overline{B}}_{\gamma u} \\ -\overline{\overline{C^+}}_{u\alpha} & -\overline{\overline{C^+}}_{u\beta} & -\overline{\overline{C^+}}_{u\gamma} & \mathbb{I} + \overline{\overline{D}} \end{pmatrix}$$

It is conceivable that for systems of high dimension, the full matrix be huge and hollow. An appropriate solving process should make use of block-matrices and of the tree-structure defining their imbedding in the global matrix.

The chosen structure applied to a TEF system is closely following the partitioning procedure. "Families" of cells are defined, gathered themselves in super-families, following the recursive partitioning of the system. The cells are the leaves of a tree corresponding to the partition in families. Subsets of transfers linking the cells can be associated with each family in an automated mode. It can be shown that their exists an isomorphism between that tree-structure and the imbedding of the block-matrices in the overall system matrix. Hence, that tree can be used to link to each block-matrix and solve partial algebraic systems recursively until all variables are computed. This process refers to a so-called "hyper multi-frontal method with super-relaxed nodes", see sources at the end of the article.

Before proceeding to the system analysis using the TEF, let comie back to an alternative solution of the TLS using a linear transformation as stated in 2.2. It will be the occasion to show the equivalence between the two solutions (time scheme and symbolic transformation).

2.6 Borel transformation on the Tangent Linear System

We previously mentioned that the TEF elements (vectors and matrices) are function of the time step δt. In order to show how the TEF elements exhibit the whole system dynamics, we introduce a TLS symbolic transformation. In lieu of the classical Laplace transformation, whose definition involves the complex space, we use the Borel transformation, defined as :

$$f(t) \overset{\mathcal{B}}{\rightarrow} \mathcal{B}[f](\tau) = \frac{1}{\tau}\int_0^\infty e^{-t/\tau} f(t)dt = \frac{1}{\tau}\widetilde{f}\left(\frac{1}{\tau}\right) \tag{9}$$

where $\widetilde{f}(p)$ is the Laplace transformation of $f(t)$. It is seen that the Borel variable τ is real and homogeneous to time. Applying this definition to system (4,5), one has

$$\left\{ \begin{array}{c} \mathcal{B}[\overrightarrow{\boldsymbol{\delta\eta}}_\alpha] = \overbrace{\left[1-\tau\left.\overline{\overline{\frac{\partial G_\alpha}{\partial \eta_\alpha}}}\right|_{t_n}\right]^{-1}\tau\ \mathbf{G}_\alpha|_{t_n}}^{\mathcal{B}[\overrightarrow{\boldsymbol{\delta\eta}}_{\alpha,dec}]} + \overbrace{\left[1-\tau\left.\overline{\overline{\frac{\partial G_\alpha}{\partial \eta_\alpha}}}\right|_{t_n}\right]^{-1}\left.\overline{\overline{\frac{\partial G_\alpha}{\partial \varphi}}}\right|_{t_n}}^{\overline{\overline{\mathcal{F}}}}\mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}] \\ \mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}] = \sum_\beta \left.\overline{\overline{\frac{\partial f}{\partial \eta_\beta}}}\right|_{t_n}\mathcal{B}[\overrightarrow{\boldsymbol{\delta\eta}}_\beta] + \left.\overline{\overline{\frac{\partial f}{\partial \varphi}}}\right|_{t_n}\mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}] \end{array} \right. \tag{10}$$

this is obtained because $\mathcal{B}[\partial f/\partial t] = (1/\tau)\mathcal{B}[f]$. After eliminating cell variables from the first equation into the second equation, the following equivalent system is obtained :

$$\left\{ \begin{array}{l} \mathcal{B}[\overrightarrow{\boldsymbol{\delta\eta}}] = \mathcal{B}[\overrightarrow{\boldsymbol{\delta\eta}}_{dec}] + \overline{\overline{\mathcal{F}}}\,\mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}] \\ \left[1+\overline{\overline{\mathcal{C}}}\right]\mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}] = \mathcal{B}[\overrightarrow{\boldsymbol{\delta\varphi}}_{ins}] \end{array} \right. \tag{11}$$

this system is analog to (8), but here τ is the dynamic variable instead of δt. This is the main property that explains the link between the forward time resolution and the dynamical characteristics of the system. This link between system (8) and (11) is obtained thanks to a property of the Crank-Nicolson time scheme, that is true for small t :

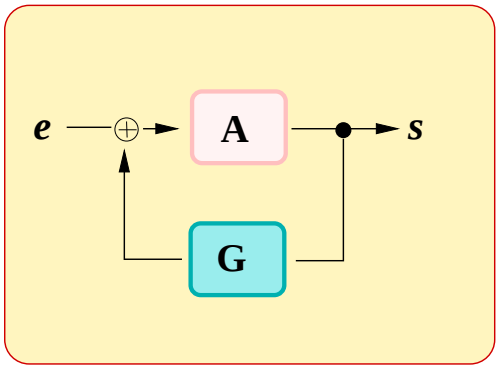
$$\delta\eta(t) \sim 2\cdot\mathcal{B}(\delta\eta)\left(\frac{t}{2}\right) \tag{12}$$

As a result, when the TLS is solved for finite increments of δt, it can be formally identified with the Borel transformation of the TLS. Said differently, the TEF elements, as functions of τ, are solutions of the TLS. The trajectory is just defined for small values of the Borel variable. In the sequel, we shall make no more distinction between the two equivalent forms. We now examine how the TEF structure of the system can bring some information related to the intelligibility of systems behaviour.

3 TEF and system analysis

The problem of obtaining some intelligibility from numerical models involves a lot of effort in numerical fluid dynamics or climate modelling in particular. The

post-treatment of simulation can reach same complexity as the simulation itself. Leaving aside the problem of analysing the spatial structures of simulated fields of parameters, we shall take as an illustration a classical type of analysis on how a system is regulating, dumping or diverging under perturbation. The concept of **feedback gain** is currently applied to explain the response of climate to forcing. Such tentative is still controversial today, because it addresses a theoretical problem that was not in the original concept definition. The concept of feedback originates in



$$s(p) = H(p)e(p) = \frac{A(p)}{1 - A(p)G(p)}e(p) \qquad (13)$$

FIG. 1 – Elementary feedback loop and associated transfer function.

Electric Circuit Theory (Bode 1945). The feedback gain is defined as a function of a symbolic variable (Laplace or Fourier transformation). Considering the input–output circuit of Fig. 1, the derivation through G takes the output signal s as amplified by A from entry, apply G, and is added to the entry e. The transfer function H directly gives the output signal when the input is given. Hence for system A, the so called feedback gain G allows to shape up signals, reduce noise, etc. Such a feedback loop is though as an active circuit built up from well known linear components. More complex circuits can be analysed in terms of arrangement of the elementary feedback-loop of Fig. 1.

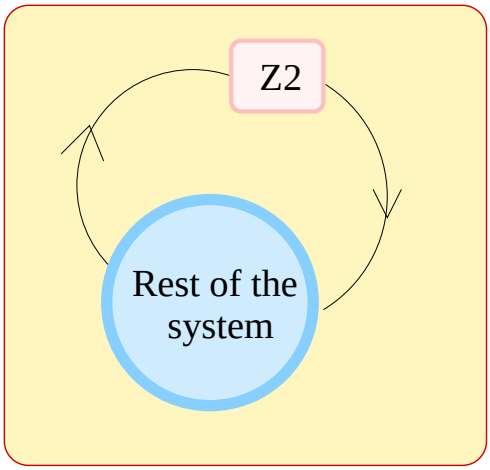
The same concept is nowadays used in a very different theoretical frame, and appears more as a conceptual analysis applied to systems that are difficult to analyse in terms of elementary components, and usually non-linear. The tentative extension of the original feedback concept emphasises on the need for building analysis tools devoted to complex systems. We show in the following that the rigorousness of the original concept can be restored for non-linear systems and in the very general frame of numerical modelling.

Dynamic feedback gain using the TEF

To be illustrative, let consider a macro-economic model, the one of IMACLIM-PROJ (cf sources). Without having to enter into details, the model is a national compatibility input-output table reduced to six sectors (oil, transportation, energy, residential housing, commercial sector and a composite sector). The model gives the response of the economic system to an exogenous increase of the oil price.

It was built to reproduce global equilibrium between householder consumption, industrial production, and employment. The system dynamics is driven by the level of consumption allowed by the prices and the wages, the repartition of consumption, and intermediate production as given by a constant Leontiev matrix. For instance, whenever the price of a product is increasing, it can be substituted by a less expensive one to optimise economic efficiency.

A more general macro-economic model may quickly reach a hight level of complexity, as the number of branches and products is more than a hundred for each country. Numerous parts of the model are also non-linear. As a result, the obtained trajectory of internal variables can be used to check for the level of realism reached by the model. It does not inform on how and why it is behaving as it does, particularly when submitted to a shock (as oil-crises, tsunamis, hurricanes). Let see how the feedback gain can help in understanding model behaviour. Take one variable of the system, which is equivalent to isolating a process associated to this variable. This operation has to be associated to a feedback loop, because the feedback is measuring the effect of opening that loop when analysing its response to a perturbation, as shown in the figure :



$$\begin{cases} \delta p + \mathbb{C}_{px}\,\overrightarrow{\delta x} = \delta p_{ins} \\ \mathbb{C}_{xp}\,\delta p + (1 + \mathbb{C}_{xx})\,\overrightarrow{\delta x} = \overrightarrow{\delta x_{ins}} \end{cases} \tag{14}$$

FIG. 2 – closed loop : only one variable (p) of the system is remaining after the elimination process.

The figure explicits how is the system acting on the price of transportation (p). The corresponding system of equations is (14). In this system, $\overrightarrow{x}$ represents the set of all system variables but p. Matrix $\mathbb{C}_{px}$ is a row vector and $\mathbb{C}_{xp}$ is a column vector. One may notice that the first equation is the transfer equation giving the price when $\overrightarrow{x}$ is representing all other variables. Let us explicit what is the effect of opening the loop :

$$[\text{Price of transportation } p] \rightarrow [\text{Rest of the system}] \rightarrow [p]$$

One eliminates the vector of variables $\overrightarrow{\delta x}$ in the price-equation p :

$$\left(1 - \mathbb{C}_{px}(1 + \mathbb{C}_{xx})^{-1}\mathbb{C}_{xp}\right)\delta p = \delta p_{ins} - \mathbb{C}_{px}(1 + \mathbb{C}_{xx})^{-1}\overrightarrow{\delta x_{ins}} \tag{15}$$

open loop : the loop is open after the transfer p, which hence cannot influence the rest of the system. This is formally done by zeroing the raw vector-matrix $\mathbb{C}_{xp}$ (the system becomes insensitive to perturbations from p).

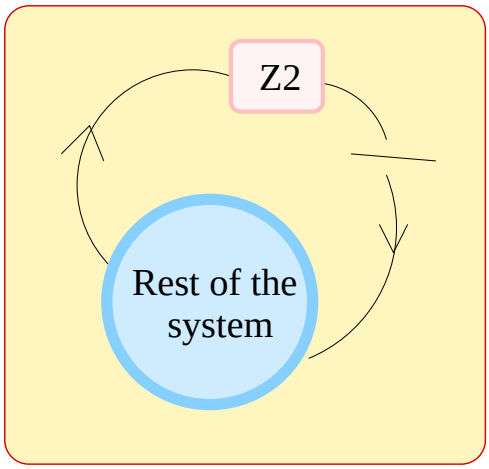


FIG. 3 – The open-loop system is obtained zeroing all matrices that allowed the rest of the system to react to a perturbation in p.

notice that the term $\mathbb{C}_{px}(1 + \mathbb{C}_{xx})^{-1}\overrightarrow{\delta x_{ins}}$ at the right of (15) is unchanged. When the loop is closed again, the original equation is restored :

$$\boxed{(1 - g)\delta p = \delta \acute{p}_{ins}}$$

where $g = \mathbb{C}_{px}(1 + \mathbb{C}_{xx})^{-1}\mathbb{C}_{xp}$, and $\delta \acute{p}_{ins} = \delta p_{ins} - \mathbb{C}_{px}(1 + \mathbb{C}_{xx})^{-1}\overrightarrow{\delta x_{ins}}$. It can be shown that this function is independent of the algebraic manipulation used to eliminate all other variables than p (the ZOOM tree-structure).

The term g is called the "feedback gain" of the system for the price of transportation. It is a function of τ (or equivalently of δt) which gives the linear response of the system when the price is perturbed (at time t on the trajectory). In our model, we found an annual gain ($\tau = 1$ year) of 15%, which has the meaning that, the system responds with an amplification of the initial perturbation.

Another result with the same model shows that for the wages of the composite sector, the system feedback leads to a gain of 50% per year. Suppose that one injects an increase of one unit to the wages in that sector, the wages will undergo an extra of 1.5 units by the end of the year after (looks like a Keynesian model). So clearly, a procedure such as the TEF allowing to explicit feedback gains do provide important information about system behaviour. One might go further, and ask why is the

system reacting in such manner. One can for example "freeze" a bunch of cells or transfers and compare the two gains, to look for the elements of the system that contribute principally to the feedback. Another possibility would involve coupling matrix analysis that consists in isolating two variables instead of one for feedback analysis.

As conclusion of this section, when comparing the proposed TEF method to the original one of electric circuit theory in looking for feedback gain, we obtained here a method to evaluate feedback gain numerically along the trajectory of non-linear systems. Whatever the complexity of a model, it is easy to analyse feedback loops.

Computational implementation
The ZOOM software

Generalities

ZOOM (Zone Organised Optimal Modelling) is the software implementation of the TEF : it computes time dependent or stationary simulations for the models which have been partitioned into cells and transfers, and provides as well, Borel transform functions devoted to system analysis [1]. Programming a model proceeds in two steps :

(**1**) The first step is the description of the different objects in which the model is partitioned. To each class of cells and transfers corresponds one model (a set of equations) which is programmed in "a processor". Processors are written in extended Fortran, so that the connections can be managed. Same processors can be used in different kinds of models so that in general, the writing of new models does not generally need more than the reprogramming of a few processors. Therefore, one can expect that if Zoom is used in economics, the accumulation of a number of processors will greatly ease the writing of new models.

(**2**) The second step is the the description of the global structure of the system, that is how cells are gathered into families, what transfers are connecting cells etc. Here also, a pseudo-Fortran programming language is used, with some facilities as :

- (i) object creation instructions ;
- (ii) connecting instructions ;
- (iii) instructions allowing the grouping of cells in families and the arrangement of families in a tree structure ;

In addition, as for any model, one has to specify initial state conditions (cells only), give value to processors parameters, choose among particular graphic output standards. By default, all variables are available as output (see ZOOM manuals).

When ZOOM is run, the first task is to connect the objects and reserve dynamic memory to store all matrices and vectors as prescribed by the TEF. Each processor is then called. The cell and transfer processors provide the necessary elements for the computation of the TEF system, so that the system can be solved each time step. Graphic output are stored in the same tree-structure as defined in the second step of programming. The graphic processing (called VIZU) is an extension of the CERN PAW code. Finally, VIZU allows for some post-processing to compute coupling matrices and feedback gains.

Processors

Processors are programmed to provide the TEF vectors and matrices needed by the TEF.

[1] A simplified version that can handle a few hundreds of equations, using the same TEF principles is also available, which includes automatic sensitivity analysis, adjoint modelling etc. see text : *Mini_Ker et Adjoint : "un résumé", and the English version of the Mini-ker manuals.*

Cells processor :

The state evolution of a cell is given on one time step by :

$$\overrightarrow{\delta\eta} = \overrightarrow{\delta\eta}_{dec} + \overline{\overline{F}}\overrightarrow{\delta\varphi}$$

where vector $\overrightarrow{\delta\varphi}$ represents a subset of the system transfers - the ones that are connected to each particular cell (α). The information of transfer variables comes with a connecting object. The cell processor can thus compute $\overrightarrow{\delta\eta}_{dec}$, and $\overline{\overline{F}}$.

- On entry, one has :
  - the time step value δt,
  - initial state variable values of the cell, the ones of the connected transfers and eventual outside sources or independent boundary conditions ($\overrightarrow{\phi_0}$).
- On output, one gets :
  - the decoupled evolution vector $\overrightarrow{\delta\eta}_{\alpha dec}$
  - along with matrix $\overline{\overline{F}}$ that gives the sensitivity of state variables to change in the connecting transfers.

Transfer processor :

Transfer variables are associated to static constraint equations that links them to connecting cells. Contrary to cell processors, transfers are also dependent from other transfer variables in general. The other transfer has to be connected to the set of cells that are also connected to the transfer at work.

$$\overrightarrow{\phi} = \overrightarrow{f}\,(\overrightarrow{\eta_{\alpha 0}}, \overrightarrow{\phi_0})$$

A transfer processor compute matrix $\overline{\overline{C^+}}$ which, analog to cells, give the sensitivity of the transfer vector to a perturbation of the connected cell variables (for the current time-step).

- On entry, one has :
  - the time step value δt,
  - initial values of connected objects : $\overrightarrow{\eta_0}$ for cells, $\overrightarrow{\phi_0}$ for eventual other transfers.
- On output, the processor provides :
  - the vector of transfer variables
  - and matrices $\overline{\overline{C^+}}$ and $\overline{\overline{D}}$.

the fact that transfers are inter-depending between themselves implies that an implicit system has to be solved to compute transfers. This is done in ZOOM at the beginning of each time step.

CONCLUSIONS

The TEF procedure of building models, as the result of partionning and re-coupling a system, gives rise to analysis of well-defined objects. The structure of the partition allows to study the inter-influence between parts of a system. Every objetc of the system has to provide TEF elements that are used to solve the overall system of equations. The elementary vectors are interpreted in terms of intrinsc behavior of the objects, while Jacobian matrices are representing local sensitivity operators.

Hence, one has the general view that explains how our original programm can be accomplished. The solution of the partictionned system has the consequence of explicitating the partial derivative chain rules that propagates local sensitivities troughout the system. Once a system reduced after eliminating a set of variables, the remaining variables keep track of their coupling through the eliminated variables. As a consequence, sensitivity to perturbations, error or noise, propagates through the system and can be analysed. The satisfactry exploitation of such facilities leads to numerous difficulties, considering in particular non-linear systems. We have shown how to characterize the local dynamics of a system using the Borel transform of feedback function. More generally, the presence of the Jacobian matrices are sufficient to automatically get the adjoint model, and hence allow to analyse the sensitivity of non-linear models and to solve optimalisation problems. Forward sensitivity also shares the same ste of Jacobian matrices, and can be used to study non-linear feedback gain as shown in the Mini-ker manuals.

Our collaboration is on its way to use and test different techiques to solve problems in a diversity of scientif fields (Engineering Sciences, Econmy and Climate, and also Biology modelling). More draft-papers are available on this web site).

Sources :

http ://www.lmd.jussieu.fr/zoom/documents.dir

- on Borel : article submit to the Journal of Atmospheric Sciences (2005)

- on the IMACLIM-PROJ model : rapport d'avancement ZOOM-ProjTES (2002) on line

- see also TEF-ZOOM seminar Volume 1, TEF, EDH, L'Agelonde, 21-25 octobre 1991 (soon on line).

- Thèses de Stéphane Blanco et Jean-Louis Joly (on request).

- on ZOOM tree-structure : Liu J.W.H. 1992 : The multi-frontal method for sparse matrix solution : Theory and Practice. *Society for Industrial and Applied Mathematics Review*,**34**, 82-109. Voire aussi manuel EDH (soon on line).